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Entropía de entrelazamento e holografía

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Abstract

In 1935 Einstein, Podolsky and Rosen discovered an unexpected and apparently paradoxical property of quantum mechanics that they called entanglement. Nowadays it is a fundamental feature of the theory and a cornerstone in quantum information theory, quantum computing and many other areas of science. The aim of this work is to develop a tool, namely the von Neumann entropy, to detect and quantify entanglement in a system.

The approach we will follow will be to successively introduce concepts and join them with the previous ones. More precisely, we will begin with entropy, then entanglement and right after show their connection. Lastly, we briefly explain the AdS/CFT correspondence and, with the help of a conjecture by Ryu and Takayanagi, link it with entanglement and use it calculate entropies.

Resumen

En 1935 Einstein, Podolsky y Rosen descubrieron una propiedad inespeperada y aparentemente paradógica de la mecánica cuántica a la cual llamaron entrelazamiento. Hoy en día es una característica fundamental de la teoría y una piedra angular en teoría de la información cuántica, computación cuántica y muchas otras áreas de la ciencia. El objetivo de este trabajo es desarrollar una herramiento, en concreto la entropía de von Neumann, para detectar y cuantificar entrelazamiento en un sistema.

El enfoque que seguiremos será el de introducir conceptos sucesivamente y relacionarlos con los anteriores. En concreto, empezaremos con la entropía, después entrelazamiento y de seguido mostraremos su conexión. Por último, explicaremos brevemente la correspondencia AdS/CFT y, con la ayuda de la conjectura de Ryu y Takayanagi, la relacionaremos con el entrelazamiento y la usaremos para calcular entropías.

Resumo

En 1935 Einstein, Podolsky e Rosen descubriron unha propiedade inespeperada e aparentemente paradóxica da mecánica cuántica á cal chamaron entrelazamento. Hoxe en día é unha característica fundamental da teoría e unha pedra angular en teoría da información cuántica, computación cuántica e moitas outras áreas da ciencia. O obxectivo deste traballo é desenvolver unha ferramenta, en concreto a entropía de von Neumann, para detectar e cuantificar entrelazamento nun sistema.

O enfoque que seguiremos será o de introducir conceptos sucesivamente e relacionalos cos anteriores. En concreto, comezaremos ca entropía, despois entrelazamento e de seguido mostraremos a sua conexión. Por último, explicaremos brevemente a correspondencia AdS/CFT e, coa axuda da conxetura de Ryu e Takayanagi, relacionaremola co entrelazamento e usarémola para calcular entropías.

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1 Introduction

Entanglement is one of the key features of quantum physics, with no analogy or similarity in classical theories. In the latter, if one is able to describe[1] a given system it is always possible to describe any part of it: knowing the state of the system is equivalent to knowing the state of each of its parts (whatever division we have made). On the other hand, in quantum physics we can have a full description of a system as a whole which fails to describe unambiguously a part of the system. A system with this behavior is said to be **entangled**.

When two particles are entangled, measurements of physical properties performed on them are found to be perfectly correlated. For example, if we generate two particles such that their total spin is zero (more technically, such that their composite (non normalized) state is of the form $|\psi\rangle = |\uparrow\rangle_1 |\downarrow\rangle_2 + |\downarrow\rangle_1 |\uparrow\rangle_2$, where subscripts 1 and 2 refer to the two different particles), and measure the spin of one of the particles, we will *automatically* know the spin of the other particle. This is, if we perform a measure that tells us that particle 1 has spin up, then we know that the state of the whole system will *collapse* to $|\uparrow\rangle_1 |\downarrow\rangle_2$, and therefore the spin of particle 2 must be down. Now, these two particles may be really far from each other, they may even be *causally disconnected*; however, as we have just argued, we can get instantaneous information about the second particle by performing a measurement on the first, violating the *local realism* view of causality. This bizzare property, which Einstein called a"spooky action at distance", was exposed in 1935 by Einstein, Podolsky and Rosen in a famous paper which became known as the EPR paradox [1]. In it they explained their view that the description of reality provided by quantum mechanics could not be complete. However, this and other counterintuitive predictions of quantum mechanics have been verified experimentally, and nowadays they are commonly accepted. The resolutions of the EPR paradox had important implications in the interpretation of quantum mechanics. Some interpretations of quantum mechanics state that the effect of the measure occurs instantaneously, while others argue that there is no "effect" at all; but, of course, both agree that, although entanglement produces correlation between measurements, no transmission of information at speeds greater than the speed of light is possible. The EPR paradox reveals just one of the deep and unexpected properties arising from entanglement.

The aim of this work is to use entropy (although not the thermal entropy that we are used to from Thermodynamics) to detect and quantify entanglement in a system. In section 2 we will introduce the so called *von Neumann entropy*, give a clear physical interpretation to it and state some of its mathematical and physical properties. In section 3 we will give a proper definition of entanglement and show the profound connection between this quantum property and the just defined mathematical entropy. We will continue in section 4 by extending this formalism from ordinary quantum mechanics to quantum field theories, where we will see the causal structure of space-time has important implications. Lastly, in section 5 we will present holography and the AdS/CFT correspondence, a duality between quantum physics and gravity theories. With this tool

[1]By *describing* we mean, in quantum terms, to know the (quantum) state of the system.

and using the Ryu-Takayanagi (RT) conjecture we will transform entanglement entropy from an algebraic quantity to a geometrical one. This will not just be a much easier method to calculate entanglement entropies but also another indication of the great connections between quantum physics and gravity. In fact, RT provides such a direct link between spatial entanglement and the dual geometry that it has been suggested that entanglement is the basic property of the field theory from which the dual gravitational space-time emerges.

2 Von Neumann entropy

Let us consider a quantum system defined by a **density matrix** ρ. This is, ρ is an operator fulfilling [2]

$$\rho^\dagger = \rho, \qquad \rho \geq 0, \qquad \mathrm{Tr}\,\rho = 1\,. \tag{1}$$

We define the **von Neumann entropy** as:

$$S(\rho) := -\,\mathrm{Tr}\left(\rho \ln \rho\right)\,. \tag{2}$$

Remembering that the expected value of an operator $\mathcal{O}$ is given by $\langle \mathcal{O} \rangle_\rho = \mathrm{Tr}\left(\mathcal{O}\rho\right)$, we see that the von Neumann entropy is nothing but the expected value of the operator $-\ln\rho$. However, this operator does not correspond to any physical observable and therefore it is hard to interpret it. Thus, (2) does not provide any intuition about von Neumann entropy. Nevertheless, using the properties of a density matrix we can associate a very clear physical meaning to this quantity. First of all, the first property in (1) guarantees the diagonalization of ρ, so that

$$\rho = \sum_a p_a \left|a\right\rangle \left\langle a\right|\,. \tag{3}$$

Thereby, (2) can be rewritten as:

$$S(\rho) = -\sum_a p_a \ln p_a\,. \tag{4}$$

The two last properties in (1) allow us to identify $\{p_a\}$ as a probability distribution (physically, p_a is the (classical) probability of finding our system in the state $|a\rangle$). Since $0 \leq p_a \leq 1$ for all a, the von Neumann entropy is a sum of non-negative numbers and we can trivially infer:[2]

$$S(\rho) \geq 0\,. \tag{5}$$

This inequality is saturated when all the terms in the sum are zero, which is equivalent to $p_a = \delta_{aa_0}$. This is,

$$S(\rho) = 0 \Longleftrightarrow \rho = \left|a_0\right\rangle \left\langle a_0\right|\,. \tag{6}$$

We say that a system with a density matrix as the previous one (i.e., a projector) is **pure**; otherwise we say it is **mixed**. Pure systems can be described using the usual formalism of

[2]We define, by continuity, $0 \cdot \ln 0 = 0$.

quantum mechanics, taking a vector in a Hilbert space as a representative of a system. In this case, the state of our system is completely known. On the other hand, mixed systems (which cannot be treated with the usual formalism) present a probabilistic nature: we cannot know with precision the state of the system but only the probability of it being in each of its possible states. Thus, (6) gives us the first physical interpretation of the von Neumann entropy: it is a detector of our ignorance about a system. In fact, although we will not do it here, it can be argued that it not only *detects* ignorance but it also *quantifies* it.

2.1 Joint systems

Let us now consider a joint system AB, whose Hilbert space $\mathcal{H}_{AB}$ may be thought of as the tensor product of both subsystems' Hilbert spaces:

$$\mathcal{H}_{AB} \cong \mathcal{H}_A \otimes \mathcal{H}_B \,. \tag{7}$$

If ρ_{AB} is the density matrix of the whole system, we define the **reduced density matrix** associated to subsystem A as

$$\rho_A := \mathrm{Tr}_B\, \rho_{AB} \,, \tag{8}$$

where $\mathrm{Tr}_B \equiv \mathrm{Tr}_{\mathcal{H}_B}$. Our goal is to construct a quantity that detects correlation between the subsystems. **Subadditivity** and **extensivity** of von Neumann entropy[3]

$$S(AB) \leq S(A) + S(B) \,, \tag{9}$$

$$S(AB) = S(A) + S(B) \Longleftrightarrow \rho_{AB} = \rho_A \otimes \rho_B \,, \tag{10}$$

suggest defining the quantity

$$I(A:B) = S(A) + S(B) - S(AB) \tag{11}$$

as a correlation detector. $I(A:B)$ is called the **mutual information** [3]. Using (9) we have

$$I(A:B) \geq 0 \,, \tag{12}$$

and (10) tells us that the equality is given when $\rho_{AB} = \rho_A \otimes \rho_B$. This last condition is the generalization of the factorization of a density function $p_{XY}(x,y) = p_X(x)p_Y(y)$ to this new density matrix formalism and therefore it is saying that both systems are independent. Thus, we can conclude that mutual information *detects* correlation between subsystems (again, it also *quantifies* it). Lastly we present here the so called **Araki-Lieb inequality**

$$|S(A) - S(B)| \leq S(AB) \,, \tag{13}$$

which will be used in the next section.

[3]From now on we will use the notation $S(\rho_X) \equiv S(X)$.

3 Entropy and entanglement

Let us consider again a joint system AB and let ρ_{AB} be its density matrix. We say the system is **separable** or **clasically correlated** if we can write

$$\rho_{AB} = \sum_i p_i \, \rho_A^i \otimes \rho_B^i \, , \tag{14}$$

where ρ_A^i and ρ_B^i are density matrices for A and B respectively and $\{p_i\}$ is a probability distribution.[4] Otherwise we say the system is **entangled**.

3.1 Pure systems

We begin our discussion analyzing the case in which our system is pure, this is,

$$\rho_{AB} = |\phi\rangle \langle\phi| \, , \tag{15}$$

and consequently we can describe it in the usual manner with the vector $|\phi\rangle \in \mathcal{H}_{AB}$. In this case the system ρ_{AB} will be separable if and only if we can write $|\phi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$, $|\phi_A\rangle$ and $|\phi_B\rangle$ being states in $\mathcal{H}_{\mathcal{A}}$ and $\mathcal{H}_{\mathcal{B}}$. We say that a vector which can be written in the previous way is **factorizable**. We then see that, for pure systems, the separability property (defined for density matrices) is *equivalent* to the factorization property (defined for vectors).

We will now characterize pure entangled systems. Using a singular value decomposition we can find sets $\{|\phi_A^i\rangle\}$ and $\{|\phi_B^i\rangle\}$ of orthonormal vectors in $\mathcal{H}_A$ and $\mathcal{H}_B$ such that

$$|\phi\rangle = \sum_i \sqrt{p_i} \, |\phi_A^i\rangle \otimes |\phi_B^i\rangle \, , \tag{16}$$

A straightforward calculation shows that

$$\rho_A = \sum_i p_i \, |\phi_A^i\rangle \langle\phi_A^i| \, , \qquad \rho_B = \sum_i p_i \, |\phi_B^i\rangle \langle\phi_B^i| \, . \tag{17}$$

Therefore, the state $|\phi\rangle$ is factorizable if and only if ρ_A is pure, which using (6) is again equivalent to $S(A) = 0$ (of course, all this is also valid for subsystem B). Summarizing the previous analysis, we have found that:

$$\boxed{\text{If } \rho_{AB} \text{ is pure,} \qquad \rho_{AB} \text{ is entangled} \iff S(A) \neq 0 \, .} \tag{18}$$

[4] This type of density matrices (convex combinations of product density matrices) are precisely those that can be obtained starting from product states $\rho_{AB} = \rho_A \otimes \rho_B$ and performing LOCC (*Local Operations and Classical Communication*, which basically are local operations performed on a part of the system and communicated classically to other part of the system, which again can perform a local operation based on the information received) [4].

The previous equivalence justifies the expression **entanglement entropy**, since von Neumann entropy *detects* entanglement. Moreover, for a pure system we have $S(AB) = 0$ and, using (13), we get $S(A) = S(B)$. Therefore we can write the mutual information as

$$I(A : B) = S(A) + S(B) - S(AB) = 2S(A)\,. \tag{19}$$

Since quantum entanglement is a type of correlation and mutual information quantifies correlation, we conclude that (entanglement) entropy *quantifies* entanglement.

This link between entanglement and entropy is valid only for pure systems, as we will see shortly. Nonetheless, the expression *entanglement entropy* is used also for mixed systems, the reason being explained in section 3.3.

3.2 Mixed systems

Once we allow our system to be in a mixed state, the previous interpretation of entropy as a measure of entanglement is lost. In a mixed system, entropy can be non zero even in a factorized density matrix $\rho = \rho_A \otimes \rho_B$, as we can see considering the following product[5]

$$\rho = \left(\sum_a p_a \left|a\right\rangle \left\langle a\right|\right) \otimes \left(\sum_{a'} p_{a'} \left|a'\right\rangle \left\langle a'\right|\right) = \sum_{a,a'} p_a p_{a'} \left|aa'\right\rangle \left\langle aa'\right|\,, \tag{20}$$

whose entropy is non zero if $p_a \neq \delta_{aa_0}$.

In subsection 2.1 we argued that mutual information measures correlation. This correlation includes classical and quantum correlation, the former arising from mixedness in separable states and the latter from entanglement in pure or mixed states. For example, consider the separable mixed state

$$\rho = \frac{1}{2}(\left|00\right\rangle \left\langle 00\right| + \left|11\right\rangle \left\langle 11\right|)\,, \tag{21}$$

which is a maximally classically correlated pair of qubits. Its mutual information is $I(A : B) = \ln 2$. If we now take the **Bell pair**

$$\left|\psi\right\rangle = \frac{1}{\sqrt{2}}(\left|00\right\rangle + \left|11\right\rangle)\,, \tag{22}$$

which is an entangled (pure) system, we find that $I(A : B) = 2\ln 2$. We can then see that entanglement is a stronger form of correlation than classical correlation, but it is also easy to realize that mutual information does not distinguish between both: two pairs of classically correlated qubits have the same mutual information as a single Bell pair.

The problem is then clear. Given a system with non zero mutual information (this is, a correlated system), how can we tell if there is quantum correlation in it? In other words, how can we tell if the system is entangled or not? This problem is very hard to solve,

[5] We will often use the notation $\left|a\right\rangle \otimes \left|b\right\rangle \equiv \left|ab\right\rangle$ and $\left\langle a\right| \otimes \left\langle b\right| \equiv \left\langle ab\right|$.

and it is in fact a NP-hard problem.[6] There are some sufficient conditions to conclude the system is entangled but a full characterization is not yet known.

3.3 Purification

We have just seen that, given a pure system AB, it being entangled is equivalent to a non zero entropy of one of its subsystems (which at the same time is equivalent to our subsystems being mixed). However, this result did not hold for mixed systems. In this section we come back to this case and see how there is still a deep relation between entanglement and entropy.

Let us suppose now that we start with a single system A in a mixed state; let then

$$\rho_A = \sum_a p_a \left|a\right\rangle \left\langle a\right| \tag{23}$$

be its density matrix (obviously we need $p_a \neq \delta_{aa_0}$ for the system to be in a mixed state). We define the **purification** of A as the state in $\mathcal{H}_A \otimes \mathcal{H}_A$

$$\left|\phi\right\rangle = \sum_a \sqrt{p_a} \left|a\right\rangle \otimes \left|a\right\rangle \ . \tag{24}$$

This is a pure state by definition, it is entangled because A is mixed and additionally $\rho_A = \mathrm{Tr}_A \left|\phi\right\rangle \left\langle\phi\right|$ (this notation means tracing over one of the factors in the tensor product).

The purification process enables us to draw a very important conclusion: our ignorance about a system (the fact that it is mixed), regardless its origin, is indistinguishable of that caused by entanglement in an associated pure system. More precisely, for pure systems we found that entanglement was equivalent to non-zero entropy (mixedness) of a subsystem; for mixed systems, the same relation applies if we think of these systems as subsystems of a larger "purified" one. Therefore, "entropy" and "entanglement" are interchangeable, making the expression "entanglement entropy", although pedagogical, a redundancy.

4 Entanglement entropy in quantum field theories

The main feature of a quantum field theory (from now on, QFT) is the intrisic existence of *space*: that is, operators depend not only on time (in Heisenberg's picture) but also on the position in a given spatial manifold. A pretty intuitive and simplified approach to introduce entanglement entropy in QFTs is to consider discretized spaces, which are way easier to work with and, often, describe real QFTs when we take the continuous limit.

[6] NP problems are those for which the problem instances, where the answer is "yes", have proofs verifiable in polynomial time by a deterministic Turing machine. Informally, NP-hard problems are those which are "at least as hard as the hardest NP problems" [5].

Let us then consider a lattice with points $\{x_i\}_{i\in I}$. On each point x_i there is a Hilbert space which we will denote by $\mathcal{H}_i$. The total Hilbert space can then be thought as

$$\mathcal{H} \cong \bigotimes_{i\in I} \mathcal{H}_i \,. \tag{25}$$

Given a subset $A \subset I$ we can factorize the previous expression as

$$\mathcal{H} \cong \mathcal{H}_A \otimes \mathcal{H}_{A^c} \,, \tag{26}$$

where $A^c = I - A$ is the complement of A and obviously

$$\mathcal{H}_A = \bigotimes_{i\in A} \mathcal{H}_i \,, \qquad \mathcal{H}_{A^c} = \bigotimes_{i\in A^c} \mathcal{H}_i \,. \tag{27}$$

We say that A is a **subsystem**. The factorization in (26) allows us to speak of entanglement entropy, mutual information and other concepts treated in the previous section. In the continuous limit it still makes sense to define a subsystem as a spatial region A,[7] and it is natural to think that a factorization as that in (26) will still hold, since operators in different points commute. However this is not quite right due to subtleties in the border δA of A, which is known as **entanglement surface**. It will not be proved here but even when such a a factorization is not possible one can still define the reduced density matrix ρ_A and consequently the entanglement entropy.

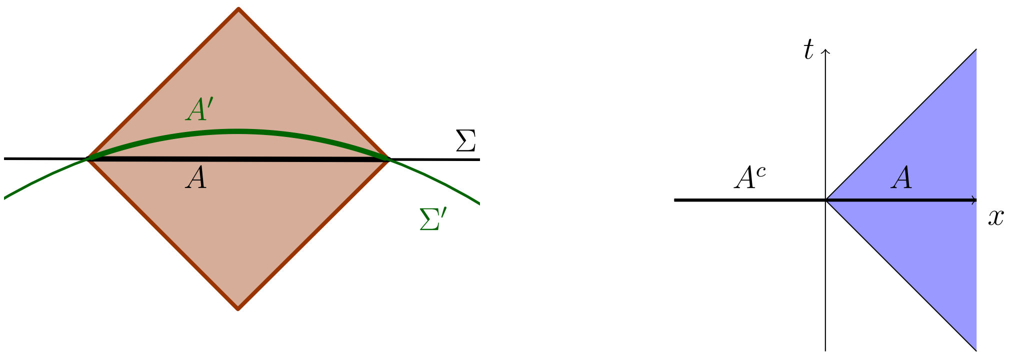
4.1 Relativistic QFTs

In a *relativistic* QFT we do not just have a space but a **space-time** $\mathcal{M}$ equipped with a causal stucture. Such a structure enables us to define the following concepts:

- Let $I \subset \mathbb{R}$ be an interval. A curve $\gamma : I \longrightarrow \mathcal{M}$ parametrized by its proper length is said to be **causal** if its tangent vector has negative or zero (pseudo-)norm at every point. We say it is maximal if $I = \mathbb{R}$ or if it starts or ends at a curvature singularity point.
- Given $S \subset \mathcal{M}$, we define the **causal domain** of S, denoted as $D(S)$, as the set of points $p \in M$ such that *every* causal maximal curve going through p intersects S.
- We say that S is **acausal** if any two points in S are not causally related.
- We say that $\Sigma \subset \mathcal{M}$ is a **Cauchy slice** if Σ is acausal and $D(\Sigma) = \mathcal{M}$.

Cauchy slices play an essential role for the following. Since Heisenberg equations, which govern the evolution of operators, must respect the causal structure, any observable can be evolved to one in a given Cauchy slice (because $D(\Sigma) = \mathcal{M}$). In this sense, operators

[7] Of course now A cannot be an arbitrary subset of the manifold, we must be careful to choose it as a subvariety of our manifold.



(a) Example of regions (marked with thicker lines) A and A' of two different Cauchy slices Σ and Σ' with the same causal domain (shown in red), where the causal structure is that of a Minkowski space.

(b) Illustration of the half-line A and its causal domain, the Rindler wedge, shown in blue. The complementary of A in the Cauchy slice $t = 0$ is labeled as A^c.

Figure 1: (a): Regions sharing the same causal domain. (b): Rindler wedge

in Σ must be *complete*, because any other observable can be written as a function of these using Heisenberg equations. Besides, the fact that Σ is acausal tells us that operators in different points of Σ commute. We can conclude that, to define subsystems, Σ plays the role that the spatial manifold played before, and a subset $A \subset \Sigma$ will have a corresponding factorization as that of (26). In fact, by the previous argument we can associate $\mathcal{H}_A$ and ρ_A not only to A but to the whole $D(A)$; or to say it another way, given another Cauchy slice $\Sigma' \subset \mathcal{M}$ and another subset $A' \subset \Sigma'$, if $D(A) = D(A')$ then $\mathcal{H}_A \equiv \mathcal{H}_{A'}, \rho_A \equiv \rho_{A'}$ and therefore $S(A) = S(A')$ (see figure 1 (a)).

4.2 Euclidean path integrals

Now we will briefly introduce the **euclidean path integral formalism**, a powerful pictorial tool to calculate transition amplitudes. It is based on the regular path integral formalism of quantum mechanics (see for example [6]), which states that the amplitude for a state $|\phi_1\rangle$ to evolve into another $|\phi_2\rangle$ in a given time T may be calculated as

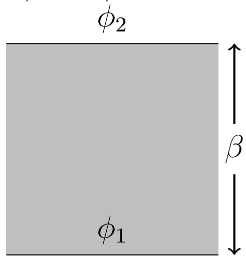
$$\langle\phi_2|\, e^{-iTH}\, |\phi_1\rangle = \int_{\phi(t=0)=\phi_1}^{\phi(t=T)=\phi_2} D\phi e^{iS[\phi]}\,, \tag{28}$$

where S denotes the action and we set $\hbar = 1$. The last integral is a sum over all possible field configurations with fixed boundary conditions at $t = 0, T$. If we make a **Wick rotation**, which changes the time variable as $t = -i\tau$, the previous expression reads

$$\langle\phi_2|\, e^{-\beta H}\, |\phi_1\rangle = \int_{\phi(\tau=0)=\phi_1}^{\phi(\tau=\beta)=\phi_2} D\phi e^{-S_E[\phi]}\,, \tag{29}$$

where S_E is the euclidean action and $\beta := iT$. The power of this change of variable is that it changes the *geometry* of our space-time, since $-dt^2$ becomes $d\tau^2$, making τ a space-like variable. The exact meaning of the integration in equation (29) depends on the topology of space. For the simple case of a space-time with $d+1$ dimensions where space is simply connected (a line (1d), a plane (2d), etc.) we can depict equation (29) by:

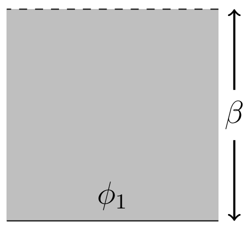
$$\langle \phi_2 | \, e^{-\beta H} \, |\phi_1\rangle =$$



$$, \qquad (30)$$

where (euclidean) time is in the vertical and space in the horizontal direction. This picture means we are integrating over $\mathbb{R}^d \times [0, \beta]$, a black (labeled) line represents boundary conditions and no line means that piece of the drawing extends to infinity. Now, in Schrödinger picture, the wavefunction of a state $|\Phi\rangle$ is completely determined by the transition amplitudes $\langle\phi|\Phi\rangle$. Therefore, we can think of $|\Phi\rangle$ as the functional $|\phi\rangle \mapsto \langle\phi|\Phi\rangle$. Then, if $|\Phi\rangle = e^{-\beta H} |\phi_1\rangle$ we can depict it as[8]

$$|\Phi\rangle = e^{-\beta H} \, |\phi_1\rangle = \int_{\phi(\tau=0)=\phi_1}^{\phi(\tau=\beta)=?} D\phi \, e^{-S_E[\phi]} =$$



$$, \qquad (31)$$

where the dashed line symbolizes we need an input for our functional. Lastly, suppose we start in some state $|Y\rangle = \sum_n y_n |n\rangle$, with $H|n\rangle = E_n |n\rangle$. Applying the operator $e^{-\tau H}$ we have

$$e^{-\tau H} \, |Y\rangle = \sum_n e^{-\tau E_n} y_n \, |n\rangle \approx e^{-\tau E_0} y_0 \, |0\rangle \,, \quad \text{for } \tau \to \infty \,. \qquad (32)$$

It follows that we can define (not nomalized) ground states by doing path integrals that extend to infinity in one direction, which we could represent as:

$$|0\rangle \propto$$

$$. \qquad (33)$$

In the sake of concreteness, let us consider as our space-time $\mathcal{M}$ a 2$-$dimensional Minkowski space with one time and one spatial coordinate. The set defined through the equation $t = 0$ is an example of a Cauchy slice. The half-line subset $A = \{(t, x) \in \mathcal{M} / \, t = 0, \, x \geq 0\}$ has a causal domain

$$D(A) = \{(t, x) \in \mathcal{M} / \, |t| \leq x\} \,, \qquad (34)$$

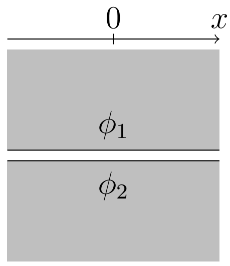
which is known as the **Rindler wedge** (see figure 1 (b)). Applying the previous tool we will now see that the vacuum $\rho = |0\rangle\langle 0|$ is a highly entangled state. Using (18) we can

[8]If we want to represent the bra $\langle\Phi|$ we just turn this picture upside down.

get an astonishing conclusion: an observer that spends his whole life inside the Rindler wedge will see his universe in a mixed state.

Using equation (33) we can represent the vacuum's $\rho = |0\rangle\langle 0|$ matrix elements as

$$\langle \phi_2 | \rho | \phi_1 \rangle = \langle \phi_2 | 0 \rangle \langle 0 | \phi_1 \rangle \propto \tag{35}$$

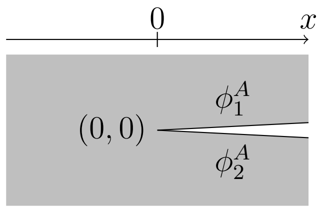


By definition we have $\rho_A = \mathrm{Tr}_{A^c}\, \rho$, which amounts to writing $|\phi_i\rangle = \left|\phi_i^A\right\rangle \otimes \left|\phi_i^{A^c}\right\rangle$ (we can think of $\phi_i^A(x)$ as the field $\phi_i(x)$ restricted to the region $x \geq 0$, and analog for A^c). Then the matrix elements of ρ_A would be:

$$\left\langle \phi_2^A \right| \rho_A \left| \phi_1^A \right\rangle = \left\langle \phi_2^A \right| \mathrm{Tr}_{A^c}\, \rho \left| \phi_1^A \right\rangle = \int D\phi^{A^c} \left(\left\langle \phi_2^A \right| \otimes \left\langle \phi^{A^c} \right| \right) \rho \left(\left| \phi_1^A \right\rangle \otimes \left| \phi^{A^c} \right\rangle \right) . \tag{36}$$

Summing over all possible fields $\left|\phi^{A^c}\right\rangle$ "glues" the top and bottom sheets along A^c in equation (35). Therefore we obtain

$$\left\langle \phi_2^A \right| \rho_A \left| \phi_1^A \right\rangle \propto \tag{37}$$



The previous picture suggests changing to polar coordinates $(r, \theta) \in [0, \infty) \times [0, 2\pi]$, where unlike normally the points $(r, 0)$ and $(r, 2\pi)$ are not identified. The matrix elements $\langle \phi_2 | e^{-\beta H} | \phi_1 \rangle$ were thought as translations in the τ directions; similarly, (37) suggests thinking of $\left\langle \phi_2^A \right| \rho_A \left| \phi_1^A \right\rangle$ as a translation in the θ direction. We have an analogy between τ and θ, or equivalently between t and $\chi := i\theta$. Therefore, we can identify

$$\rho_A \equiv e^{-2\pi K} , \tag{38}$$

where K is the generator of χ translations, just as H is the generator of t translations. This state is a *Gibbs' state* and, since $e^{-2\pi K} = \sum_k e^{-2\pi\lambda_k} |k\rangle\langle k|$, where $K|k\rangle = \lambda_k |k\rangle$, we see it is a mixed state. Therefore, we can conclude that the vacuum is an entangled state.

The last equation is telling us that, to an observer confined to the Rindler wedge, the vacuum looks like a thermal state (with respect to the (boost) generator K) at a temperature $T_\chi = 1/(2\pi)$. It is important to notice that this is not the physical temperature such an observer feels, since we must take into account the local **redshift** factor. Thus we have

$$T_{\mathrm{phys}} = \frac{T_\chi}{\sqrt{-g_{\chi\chi}}} = \frac{1}{2\pi r} , \tag{39}$$

where we have used that $ds^2 = dr^2 + r^2 d\theta^2 = dr^2 - r^2 d\chi^2$. For instance, an observer following a world-line of constant r has a trajectory in the original (t, x) coordinates of the form

$$x(t) = \sqrt{r^2 + t^2}\,, \tag{40}$$

which is the equation of a hyperbola in the (t, x) plane. This corresponds to an observer with constant acceleration $a = 1/r$, and the fact that an observer being accelerated at a constant proper rate a experiences a temperature $a/(2\pi)$ is known as the **Unruh effect** [7]. The dependence $T \propto r^{-1}$ means that the fields are very "hot" next to the entangling surface.[9] This has a clear physical interpretation: when we trace over A^c we transition from a pure to a mixed state, which makes some of the field modes to decohere, and the closer the observer is the more UV modes he can see to be decohered. It is also interesting to note that, although the physical temperature is spatially varying, the system is in perfect thermal equilibrium thanks to the varying gravitational potential (in other words, although the physical temperature T_{phys} felt by an observer was found to have an $r-$dependence, the system's temperature $T_\chi = 1/(2\pi)$ is constant).

The fact that ρ_A can be understood as a thermal state means that we can calculate its entropy as a thermodynamic entropy; that is, in this case the entanglement entropy $S(A)$ has the meaning of a real physical entropy (as seen by an observer confined to the Rindler wedge). We will use this to *estimate*[10] the value of $S(A)$ by adding up local thermal entropies. We need to take two considerations into account. First, for a field of mass m the entropy density essentially vanishes for $T < m$, since the field is frozen out. Therefore, remembering that $T \propto r^{-1}$ we can approximate $s(T(r)) \approx 0$ for $r > \zeta := m^{-1}$. In other words, what we are saying is that the field is entangled only with a neighborhood $r \leq \zeta$ of the entangling surface (in fact, this intuitive statement holds for any dimension and any entangling surface). Second, for temperatures much higher than m, by dimensional analysis we find $s(T) \propto T$. Thus, for $r << \zeta$ the entropy density diverges and so does its integral. Hence, we need (as usual) to impose an UV cutoff at $r = \epsilon$. With these previous considerations we have:

$$S(A) \approx \int_\epsilon^\infty dr s(T_{\text{phys}}(r)) \propto \int_\epsilon^\zeta dr \frac{1}{r} = \ln\frac{\zeta}{\epsilon}\,.$$

We can get a more precise estimate if we assume that the theory has an UV fixed point (a fixed point in the renormalization group flow [8]) which is a **Conformal Field Theory**. A conformal field theory (usually denoted as CFT) is a type of quantum field theory that is invariant under the *conformal group*, which is the set of transformations of spacetime that preserve angles (but not necessarily lengths). Of course, the conformal group includes all the transformations in the Poincarè group, but besides there are others such as scale transformations (in fact, for two dimensions it can be shown that scale invariance is enhanced to conformal invariance). Generally speaking, we could say conformal transformations are coordinate transformations that are a local rescaling of

[9] All the hyperbolas in equation (40) have the curve $|t| = x$ as asymptote and intersect at the x-axis at $(0, r)$, so the smaller r is the closer we are to the entanglig surface.

[10] This is an approximation, since thermal entropy densities are defined only for flat space at a *constant* temperature, whereas here we have seen that T is spatially varying.

the metric. That being said, the entropy density (per unit volume) of a CFT (at finite temperature) with central charge c is given by

$$s(T) = \frac{2\pi c}{6} T \,, \tag{41}$$

and therefore

$$S(A) \approx \int_{\epsilon}^{\zeta} dr\, s(T_{\text{phys}}(r)) = \frac{c}{6} \int_{\epsilon}^{\zeta} \frac{dr}{r} = \frac{c}{6} \ln \frac{\zeta}{\epsilon} \,. \tag{42}$$

From this equation we learn that, on one side, the entropy has a logarithmic UV divergence proportional to the central charge c and, on the other side, that it is IR-finite if and only if the field is massive.

For the case of a massless field ($m \to 0$ or equivalently $\zeta \to \infty$) we can solve this IR-divergence by considering a finite interval instead of the whole half-line; let then

$$A = \{0\} \times \left[-\frac{l}{2}, \frac{l}{2} \right] \,. \tag{43}$$

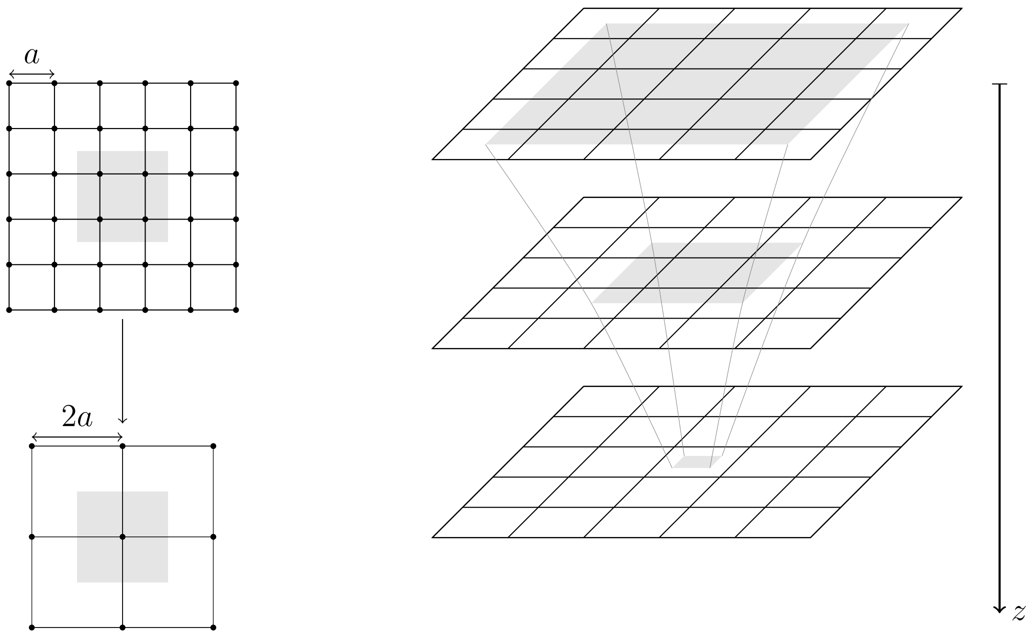
Note that, by translation symmetry, this is equivalent to considering $A = \{0\} \times [0, l]$, but we center the interval around $x = 0$ to obtain a more clear comparison with the calculation that will be made in subsection 5.2. In this case the entangling surface consists of two separate regions (the two endpoints of the interval) and the entropy, by translational symmetry, can only depend on their relative position, the length l of the interval. At first it is natural to think that the entropy cannot depend on l, since our theory is conformally invariant. However, we have seen that it does depend on the UV-cutoff ϵ, breaking the conformal symmetry and allowing the entropy to depend on l/ϵ. As we saw in (42), the divergence of the entropy near the endpoint (entangling surface) was of the form $(-c/6) \ln \epsilon$ and, since we are working with an entangling surface composed by two endpoints, we will now have a divergence of the form $(-c/3) \ln \epsilon$. As discussed before, the dependence of the entropy with ϵ automatically dictates the dependence on l, with which we finally get

$$S(A) = \frac{c}{3} \ln \frac{l}{\epsilon} \,. \tag{44}$$

If we analyze the steps and assumptions that led to the previous equation, we find it is valid for any interval of length l of any CFT with central charge c in flat space-time.

5 Entanglement entropy and holography

As we have discussed in section 3, entanglement entropy gives us a lot of physical information about a system. However, as one can guess from subsection 4.2, calculating entanglement entropies is *very* hard except in a few cases (in the previous example, one of those few cases, we were able to find an expression for the entropy using that the obtained reduced density matrix was a thermal state). The solution to this apparent problem was given in 2006 by Shinsei Ryu and Tadashi Takayanagi, who conjectured a beautiful geometrical way to calculate entanglement entropies using *holographic* techniques. To understand their proposal we first need to take a small detour and introduce the AdS/CFT correspondence.



(a) As we double the lattice spacing we substitute every four points by a single one located at their middle.

(b) If we draw several grids with the same number of points, we can see the relative size of the gray region with respect to the lattice spacing and the shrinking effect becomes clear. We see how the gray region reduces its side as z increases.

Figure 2: Kadanoff-Wilson renormalization of a lattice

5.1 AdS/CFT correspondence

The **AdS/CFT correspondence** was proposed in 1997 by Juan Maldacena [9]. It is a duality[11] between conformal field theories and quantum gravity theories in an anti-de Sitter spacetime. The field theory lives in a $D-$dimensional space-time, while the quantum gravity theory adds up an extra dimension and lives in a $(D+1)$-dimensional space-time;[12] one can think of the field theory as *living at the boundary* of its dual gravity theory, which justifies the name of **holographic** correspondence.

We will try to motivate the correspondence and this extra dimension from the **Kadanoff-Wilson renormalization group** approach. As we did in section 4, let us *discretize* our field theory and consider it as a lattice system with lattice spacing a: we can think of this parameter a as the *resolution* we have of the system (the smaller a is the more points we have access to). If we want to simplify our system we can substitute a collection of points, for example four of them, by a single one. This amounts to reducing our system to one with four times less points and lattice spacing $2a$ (see figure 2). We could repeat this process indefinitely (supossing our system has enough points) obtaining a sequence of lattices with lattice spacing $u = (a, 2a, 4a, 8a, \ldots)$. The key idea is to promote the sequence of lattice spacings u to a *continous* variable $u \in [0, \infty)$ and think of it as a **new**

[11] A duality is a non-trivial physical equivalence between two seemingly very different theories.

[12] Originally, it was formulated for $D = 4$ [9].

dimension; that is, the different lattices shown in figure 2 are actually thought of to be layers of a new space with an extra dimension.

To motivate the gravity nature of this new space let us consider a region of our lattice, as the gray one shown in figure 2. If we get two lattices with spacings $u < u'$, it is clear that the same region will appear smaller (in comparison to our only parameter, the lattice spacing) in the lattice with the biggest spacing.[13] This is telling us that our new dimension is changing the *geometry* of the space. Following Einstein's remarks on the connection between geometry and gravity, it seems natural to add **gravity** to this higher-dimensional new space. This is the main idea of the duality: given a $D-$dimensional QFT there is an associated gravity theory living in $(D+1)-$dimensions, with the extra dimension amounting to the scale at which we probe our QFT. The QFT lives at the boundary $u = 0$ (no lattice spacing, continuous theory) of its gravity dual. [14]

For an arbitrary QFT, the geometry of its associated gravity theory is usually very hard to calculate. However, for the case of a **Conformal Field Theory** it is possible to find the associated geometry. Indeed, let $(t, x_1, \ldots, x_{D-1}) = (t, \vec{x})$ be coordinates of our $D-$dimensional QFT and let z be the extra dimension. The most general metric in the $(D+1)-$dimensional space which preserves Poincaré invariance (invariance under translations, rotations and boosts) in D dimensions has the form:

$$ds^2 = \Omega^2(z)(-dt^2 + d\vec{x}^2 + dz^2)\,, \tag{45}$$

where $\Omega(z)$ is a function to be determined. We have argued that z represents the scale at which we consider the theory. Therefore, the transformation $(t', \vec{x}\,', z') = \lambda(t, \vec{x}, z)$ preserves distances and, for our theory to be conformal invariant we must have

$$ds'^{\,2} = \Omega^2(\lambda z)\lambda^2(-dt^2 + d\vec{x}^2 + dz^2) = \Omega^2(z)(-dt^2 + d\vec{x}^2 + dz^2) = ds^2\,, \tag{46}$$

which implies $\Omega^2(\lambda z) = \Omega^2(z)\lambda^{-2}$. Therefore we find

$$\Omega(z) = \frac{L}{z}\,, \tag{47}$$

where the constant L is known as the anti-de Sitter **radius**. In summary, the line element of the dual gravity theory

$$ds^2 = \frac{L^2}{z^2}(-dt^2 + d\vec{x}^2 + dz^2)\,, \tag{48}$$

is that of an AdS spacetime.

5.1.1 Counting the degrees of freedom

Once we have established this correspondence we must set the rules to translate quantities and concepts from one side to the other. The first thing we will do is match the degrees of freedom in both sides.

[13]We can think of an analogy with a camera. If we are very close to an object we will have *high* resolution of it (*small* u) and it will appear very *big*; as we move away from it, the resolution will be *lower* (*bigger* u) and the object will appear *smaller*.

[14]This correspondence was established on a firm basis in the framework of string theory. Here we are providing a quick motivation on physical grounds.

In the QFT side we must impose UV and IR regulators to avoid divergences. Let then ϵ and R be the the UV regulator (lattice spacing) and IR cutoff (system's size) respectively. If our space-time has d dimensions ($d-1$ space-like and 1 time-like) then we have R^{d-1}/ϵ^{d-1} cells, and therefore the number N^{QFT} of degrees of freedom will be

$$N^{QFT} = c_{QFT} \left(\frac{R}{\epsilon}\right)^{d-1} , \tag{49}$$

where c_{QFT} is called **central charge** and it represents the number of degrees of freedom per lattice site.

In the gravity side, the entropy of a volume is bounded by that of a black hole fitting inside the volume [10]. According to Bekenstein-Hawking formula this entropy is given by

$$S_{BH} = \frac{A_{\text{hor}}}{4G} , \tag{50}$$

where A_{hor} is the area of the event horizon and G is Newton's gravitational constant. Since the number of degrees of freedom is of the same order as the maximum entropy, we have

$$N^{AdS} \sim \frac{A_\partial}{4G} , \tag{51}$$

where A_∂ is the area of the region at the boundary $z=0$ of our $(d+1)-$dimensional AdS space. This area is divergent so we instead calculate the area of a slice at $z=\epsilon$:

$$A_\partial = \int_{\mathbb{R}^{d-1}, z=\epsilon} d^{d-1}x\sqrt{g} = \left(\frac{L}{\epsilon}\right)^{d-1} \int_{\mathbb{R}^{d-1}} d^{d-1}x \,. \tag{52}$$

Similarly as in the QFT side we must consider only a portion of the space to solve the divergence. For this we simply put our system in a box of size R, thus obtaining

$$A_\partial = \left(\frac{L}{\epsilon}\right)^{d-1} \int_{[0,R]^{d-1}} d^{d-1}x = \left(\frac{LR}{\epsilon}\right)^{d-1} . \tag{53}$$

As expected, N^{QFT} and N^{AdS} both scale the same way with the UV and IR regulators ϵ and R. For the case $d=2$ an exact calculation was made in [11] even before Maldacena's paper. In it Brown and Henneaux defined the so called *asymptotic symmetries* and found that in 3d-gravity (AdS_3) they fulfilled a Virasoro algebra with central charge c given by:

$$c_{QFT} = \frac{3L}{2G}, \qquad \text{for } d=2 \,. \tag{54}$$

We can see that it differs only by a factor 6 from what we would have gotten imposing $N^{QFT} = N^{AdS}$ with the previous order-of-magnitude analysis.

5.1.2 Correlation functions

Now we will see how to calculate correlation functions of the form $\langle \mathcal{O}(x_1)\ldots\mathcal{O}(x_n)\rangle$ in the gravity side. In field theories we can calculate them from functional derivatives of a *generating function* as follows. We perturb the lagrangian by a source term as $\mathcal{L}' = \mathcal{L} + J(x)\mathcal{O}(x) \equiv \mathcal{L} + \mathcal{L}_J$, where

$$Z_{QFT}[J] = \left\langle \exp\left[\int d^d x \mathcal{L}_J\right]\right\rangle_{QFT}, \tag{55}$$

is the generating functional. Then, the correlators can be calculated as

$$\langle \mathcal{O}(x_1)\dots\mathcal{O}(x_n)\rangle = \frac{\delta^{(n)}}{\delta J(x_1)\cdots\delta J(x_n)} \ln Z_{QFT}[J]\Big|_{J=0}. \tag{56}$$

Moving now to the gravity side, let $\phi(z,x)$ be a field in AdS and let $\phi_0(x) := \phi(0,x)$ be the boundary value of ϕ. The field ϕ_0 is identified as the source $J(x)$ for some dual operator $\mathcal{O}$ in the QFT side.[15] The AdS/CFT prescription reads [13, 14]:

$$Z_{QFT}[\phi_0] = \left\langle \exp\left[\int d^d x \phi_0 \mathcal{O}\right]\right\rangle_{QFT} = Z_{\text{gravity}}[\phi_0], \tag{57}$$

where $Z_{\text{gravity}}[\phi_0]$ is the partition function (i.e., a path integral) in the gravity theory evaluated over all functions ϕ such that $\phi(z=0,x) = \phi_0(x)$:

$$Z_{\text{gravity}}[\phi_0] = \sum_{\phi(0,x)=\phi_0(x)} e^{S_{\text{gravity}}}. \tag{58}$$

In the limit in which classical gravity dominates (note that this limit can be taken and is precisely the large N limit discussed by Maldacena [9]) we can substitute the sum by a single (renormalized) term $\exp S_{\text{grav}}^{\text{ren}}$ corresponding to the classical solution, so that according to equation (57) we have $\ln Z_{QFT}[\phi_0] = S_{\text{grav}}^{\text{ren}}[\phi_0]$ and therefore:

$$\langle \mathcal{O}(x_1)\dots\mathcal{O}(x_n)\rangle = \frac{\delta^{(n)}}{\delta \phi(x_1)\cdots\delta \phi(x_n)} S_{\text{grav}}^{\text{ren}}[\phi]\Big|_{\phi=0}. \tag{59}$$

Equation (57) is the golden rule of what could be called the "holographic dictionary", a set of prescriptions to translate and relate observables from one side of the duality to the other. Among others (see for example [12]), it includes the relation between fields in the gravity side and sources for operators in the field theory or the relation between the AdS radius and the central charge. It is important to realize how important this dictionary is, since the AdS/CFT correspondence is nothing but a *conjecture*. Thus, it is this dictionary that allows us to actually compare quantities and observables in both sides, giving physical reality to the abstract theoretical connection between theories.

For the purpose of this work, it would be convenient to find a way to calculate entropies using the duality. In the next section we will find how to do it, and we will see how, aside from its pure theoretical interest, the correspondence often turns a rather difficult problem into a relatively easy one.

[15] Strictly speaking, the source is not ϕ_0 but a regularized value, since $\phi(z,x)$ diverges as $z \to 0$ (see for instance [12])

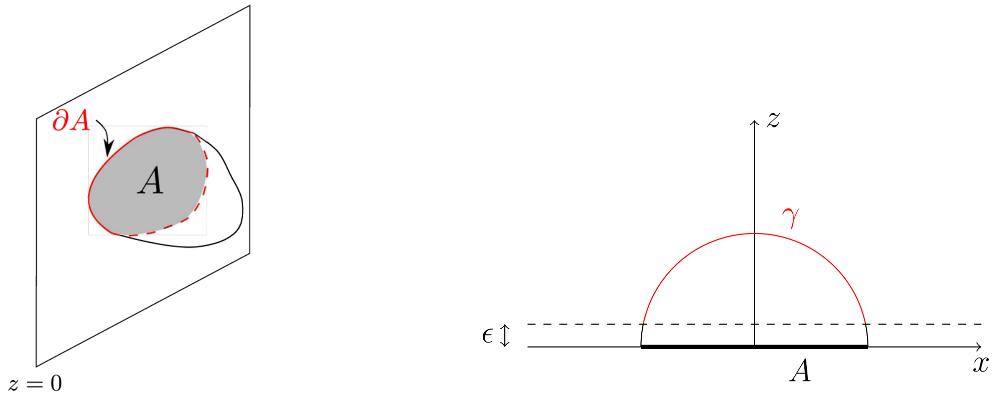


Figure 3: *Left:* We show a region A of our QFT (at the boundary $z = 0$ of the gravity theory) and an homologous surface anchored to A. Note that we are working on a Cauchy slice and therefore the time coordinate is supressed. Image obtained from [15]. *Right:* Due to divergence problems we must insert an UV regulator at $z = \epsilon$ and calculate the length of curves γ cutted at this point.

5.2 Ryu-Takayanagi conjecture

The Ryu-Takayanagi (RT) conjecture states that, given a region $A \subset \Sigma$ of a Cauchy slice (which must be either static or invariant under $t \to -t$) in our QFT, the entanglement entropy of A is given by[16]

$$S(A) = \frac{c^3}{4G\hbar}\text{area}(\gamma_A)\,, \tag{60}$$

where γ_A is a codimension-2 surface in the gravity dual theory satisfying:

1. γ_A is anchored to A.
2. γ_A is homologous to A.
3. γ_A extremizes the area of a surface fulfilling conditions 1 and 2; if there are several extremal surfaces, we choose γ_A as the one with the least area.

The first property means that γ_A "hangs" off A, as shown in figure 3 (the reason why it goes down instead of staying close to A is the prefactor $1/z^2$ in (48), which makes areas smaller for higher z). The meaning of the second one requires homology theory, a part of algebraic topology, but for our purposes let us say it is sufficient that the surface γ_A is *continously deformable* into A. Lastly, the meaning of the third property is clear.

In order to make a concrete computation and show how the RT prescription works, let us come back to the 2d-case treated in subsection 4.2 and consider a $(1+1)$d CFT. As before, we take as our Cauchy slice the region defined through $t = 0$. In it we consider the interval:

$$A = \{0\} \times \left[-\frac{l}{2}, \frac{l}{2}\right]\,. \tag{61}$$

[16]We will often set $c = \hbar = 1$ and simply write $S(A) = \frac{1}{4G}\text{area}(\gamma_A)$

We have seen that the dual geometry to a CFT is an AdS space with one extra dimensions. Since we are working at a fixed time $t = 0$, a codimension-2 extremal surface is in this case one-dimensional, i.e., a geodesic. Therefore, acoording to equation (60), to find the entropy of region A we must calculate the length of the geodesic connecting the points $(z_1, x_1) = (0, -l/2)$ and $(z_2, x_2) = (0, l/2)$ in a space with a geometry given by (see equation (48)):

$$ds^2 = \frac{L^2}{z^2}(dx^2 + dz^2)\,, \tag{62}$$

which we see is the hyperbolic plane. It is simply to realize that this length will diverge, and it is interesting to see how this divergence arises. The blow-up of the metric next to the boundary $z = 0$ forces the geodesics to hit it orthogonally. Since the length element is $ds \sim l dz/z$ we see that the divergence is logarithmic (with a certain factor). In fact, we can generalize this result for higher dimensions $D > 2$. Due to the factor $1/z^2$ in the metric, the minimal area surface will hit the boundary orthogonally. Therefore, we can approximate the near-horizon area as a slice of length ϵ hanging down orthogonally from ∂A. Thus, the leading divergence will be

$$\int_{\gamma_A} d^{D-1}x\sqrt{h} \approx L^{D-1}\int_{\partial A} d^{D-2}x\sqrt{h_{\partial A}}\int_0^{\epsilon} dz\, z^{1-D} = \frac{L^{D-1}\sigma}{(D-2)\epsilon^{D-2}}\,, \tag{63}$$

where h is the determinant of the induced metric on γ_A, $h_{\partial A}$ is the determinant of the metric on ∂A induced by the boundary metric (in our case Minkowski) and σ is the area of the entangling surface ∂A (with respect to the boundary metric). This is the gravity dual of the familiar UV divergence problem in QFT, so we proceed as usual: cut off the space-time at $z = \epsilon$. Therefore, we want to calculate the length of a curve as that in figure 3. Parameterizing the curve $(z(\lambda), x(\lambda))$ by $\lambda = z$ we find

$$\text{length}(\gamma) = \int_\gamma ds = \int_\gamma \frac{L}{z}\sqrt{dx^2 + dz^2} = 2L\int_{z=\epsilon}^{z=z_{max}} dz\frac{\sqrt{1 + \left(\frac{dx}{dz}\right)^2}}{z}\,, \tag{64}$$

where the upper integration limit z_{max} and the factor 2 reflect the fact that the curve rises and then drops down symetrically so we can just calculate half of its length. Taking the integrand as a 1d "action" and applying Euler-Lagrange equations we find (note that $\frac{\partial \mathcal{L}}{\partial x} = 0$ and therefore x is cyclic):

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\dot{x}}{z\sqrt{1+\dot{x}^2}} = C \in \mathbb{R}_+ \Longrightarrow \dot{x}^2 = \frac{C^2 z^2}{1 - C^2 z^2} \Longrightarrow (x-K)^2 + z^2 = C^{-2}\,, \tag{65}$$

which is the equation of a (semi-)circle centered at $(K, 0)$ and with radius C^{-1}, namely $x^2 + z^2 = l^2/4$ if we impose boundary conditions (this is, $K = 0$ and $C = 2/l$). Going back to equation (64) with the expression for $\dot{x}^2(z)$ obtained in equation (65) we get

$$\text{length}(\gamma_A) = 2L\int_{z=\epsilon}^{z=l/2} dz\frac{\sqrt{1 + \frac{(2/l)^2 z^2}{1-(2/l)^2 z^2}}}{z} = 2L\int_{z=\epsilon}^{z=l/2} dz\frac{1}{z\sqrt{1-(2/l)^2 z^2}}\,. \tag{66}$$

The previous integral can be computed making the change of variable $z = (2/l)^{-1}\sin\theta$; the result is

$$\begin{aligned}\text{length}(\gamma_A) &= 2L\left[\ln z - \ln\left(\sqrt{1-(2/l)^2z^2}+1\right)\right]_{\epsilon}^{l/2} \\ &= 2L\ln\left[\frac{l/2}{\epsilon}\left(\sqrt{1-\left(\frac{2}{l}\epsilon\right)^2}+1\right)\right] \\ &\approx 2L\ln\frac{l}{\epsilon}\,,\end{aligned} \tag{67}$$

where in the last approximation we have used that, at first order, $\sqrt{1-\left(\frac{2}{l}\epsilon\right)^2} \approx 1$. Plugging this result into equation (60) with $\hbar = c = 1$ we finally obtain

$$S(A) = \frac{L}{2G}\ln\frac{l}{\epsilon}\,. \tag{68}$$

Using equation (54) we get $L/(2G) = c/3$ and we verify that RT leads to the same result as (44).

We have just seen the first non-trivial check of the RT prescription to compute the entanglement entropy by holographic means. Now we will see that RT is also instrumental to prove a non-trivial statement in quantum information theory: the strong subadditivity property.

5.2.1 Proof of strong subadditivity

In subsection 2.1 we said that mutual information quantifies correlation. One of the facts that supports this statement is the so called **strong subadditivity** property, which states that

$$I(A:B) \leq I(A:BC)\,. \tag{69}$$

This non-decreasing behaviour of mutual information when we adjoin another system to A or B is saying that a system is more correlated with two partners than one (i.e., the second partner "adds" its correlation). According to the definition (11) of mutual information the last inequality can be rewritten as

$$S(ABC) + S(B) \leq S(AB) + S(BC)\,. \tag{70}$$

This inequality, which is a cornerstone in quantum information theory, has a tedious and long proof which involves many properties of traces and inequalities for convex functions [16]. However, we will now show a very simple proof using the RT-formula (60).

Let us consider the curves shown in figure 4 with minimal area (length) associated to the systems ABC (green), B (red), AB (blue) and BC (black). Let X be the length of the curve anchored to B obtained by following the black line until its intersect with the blue one, and from there the blue one. The fact that the red curve has minimal length implies that length(red) $\leq X$. Similarly, if Y is the length of the curve anchored to ABC obtained by following the blue line and then the black one, minimality implies

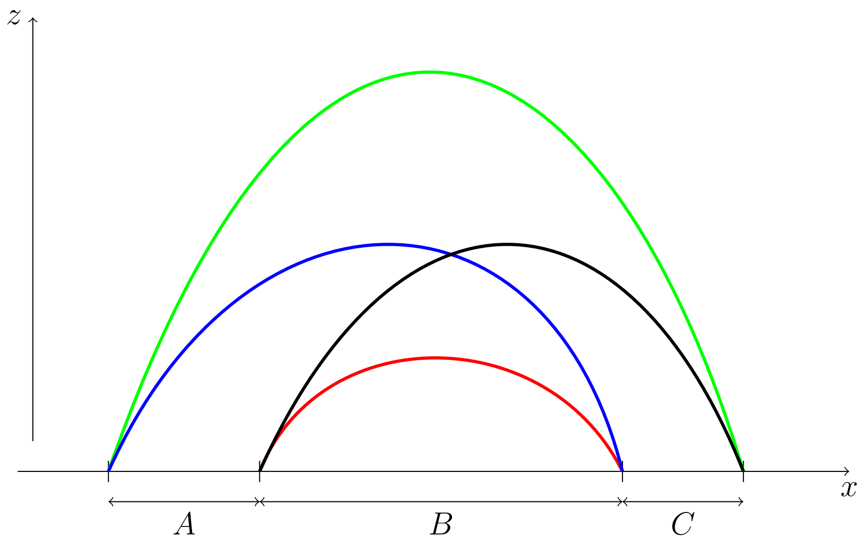


Figure 4: Graphical proof of strong subadditivity using RT.

that length(green) $\leq Y$. Therefore we have length(red) + length(green) $\leq X + Y =$ length(blue) + length(black). Using the RT-formula (60), the last statement is simply the strong subadditivity inequality.

6 Conclusions

By the end of this dissertation, I hope it has become clear the importance of entanglement in quantum physics and the power of entropy to work with this quantity.

We began defining both entropy and entanglement as independent properties, but it soon became apparent that these two quantities are deeply connected. For the case of a pure system, we saw how it being entangled was equivalent to a non-zero entropy of its subsystems. For a mixed system, it was possible for it to be non entangled but for a subsystem to have a non-zero entropy; however, in this case the purification process allowed us to think of our whole mixed system as part of a bigger, entangled one. This justified the term "entanglement entropy": entanglement in a system is equivalent to a non-zero entropy of a smaller one.

We proceeded by extending our concepts to quantum field theories. Here, we saw how the causal structure of space-time brought some interesting features. We did not need to analyze the whole space-time but only Cauchy slices, since their operators must be complete in the sense that any other operator could be written as a function of those on a Cauchy slice using Heisenberg equations. It is within these Cauchy slices that we defined subsystems, and we argued that entropy is dependent on a subsystem's causal domain and not the actual subsystem itself. To close section 4 we calculated the entropy of an easy example, primarily to show how difficult it can be.

Lastly, we introduced the AdS/CFT correspondence and exposed some entries in the dictionary that allows ys to relate quantities and observables at both sides of the duality. Specifically, we focused on the Ryu-Takayanagi conjecture, and with it we were able to calculate the entropy for the same case as in subsection 5.2 by geometrical means and, as expected, found a total agreement.

Besides all the material we have presented here, there are still many other topics and even open problems related to entanglement entropy and holography. To mention a very recent one, let us suppose we have a black hole. We will make two assumptions: first, that seen from outside it can be described as a quantum system with S degrees of freedom, with $S = A/(4G)$ being the entropy given by (50); second, that it evolves according to unitary evolution, also seen from outside. As Hawking proved in [17], black holes emit radiation, and this radiation is entangled with partners of radiation that fall into the black hole. As the black hole *evaporates*, more radiation will be emitted and, since we can only observe the exterior of a black hole, the entropy of the radiation will increase steadily until it reaches a maximum when the black hole evaporates. On the other side, the (thermodynamic) entropy of a black hole is proportional to its area [18, 19], and therefore its entropy will decrease as its evaporates. However, as the black hole's entropy decreases so do its degrees of freedom, and when it is small enough it does not have enough degrees of freedom to entangle with the emitted radiation. Therefore, we would expect that there is a certain time when the entropy of the radiation stops increasing and starts decreasing (making the shape of an upside-down V), contrary to Hawking's predictions in his famous black hole information paradox [20]. This is the so called **Page curve**, and last year the RT formula has been modified successfully to be able to reproduce this expected behavior [22, 23]. We can also see this noticing that, from the definition we have given for entropy, it is clear that $S(A) \leq \ln \dim \mathcal{H}_A$. Furthermore, in a pure state we have $S(A) = S(A^c)$, and therefore $S(A) \leq \min\{\ln \dim \mathcal{H}_A, \ln \dim \mathcal{H}_{A^c}\}$. It turns out that in the limit of large Hilbert spaces, a typical (or random) pure state saturates the previous inequality. Thus, assuming that the dynamics governing the radiation process is essentially random (except for determining the amount of radiation as a function of time and, therefore, the sizes of the black hole and radiation's Hilbert spaces) the black hole's entropy will follow a Page curve.

We emphasize on the profound connection that RT makes between spatial entanglement and the dual geometry. As it has been suggested, the RT prescription seems to insinuate that entanglement is the property of the field theory from which the dual gravitational space-time emerges. Whether the AdS/CFT correspondence turns out or not to be a suitable description of quantum gravity, I hope in this work we have reinforced the great link between the two major theories of modern physics.

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